TABLE 2. INTERFACE DEFLECTION AND RESULTS OF NUMERICAL INTEGRATIONS FOR EXAMPLES IN FIGURE 2

λ	Fluids	$f_1'(0)$	$f_1''(0)$	α	β	η_1^o
0	A	0.381	0.250	1.079	1.934	−1.079
	B	0.101	0.319	1.660	0.940	−1.660
	C	0.317	0.272	1.114	1.416	−1.114
1/3	A	0.546	0.186	0.802	0.678	−1.41
	B	0.375	0.252	1.090	0.234	−3.38
	C	0.503	0.211	0.787	0.452	−1.57

same order with Newtonian liquids of equal densities, but the effect on interface deflection is roughly the same as for Newtonian fluids. For this example, increasing m_1 by a factor of 9.3 increases the interface velocity by a factor of 3.8 and increases the interface deflection y^o at any axial distance x by a factor of 2.7. To produce the same changes in interface velocity and deflection in Newtonian fluids of the same density, with $\mu = m^{1/n}$ initially, it is necessary to increase μ_1 by factors of 54 and 3.7, respectively.

One would expect comparison of these shear-layer solutions with experiment to show better agreement than is found in the case of jets (*8*), because the power-law approximation is best in the center of the shear layer where the shear rate is largest, whereas it is known to be invalid on the centerline of a jet where the velocity gradient vanishes.

ACKNOWLEDGMENT

This research was supported by the Advanced Research Projects Agency (Project DEFENDER) under Contract DA-31-124-ARO-D-257, monitored by the U.S. Army Research Office, Durham.

NOTATION

f = dimensionless stream function
m, n = parameters characterizing a power law fluid
u, v = fluid velocity in x and y directions, respectively
U_1 = free-stream velocity of higher velocity fluid
U_2 = free-stream velocity of lower velocity fluid
x = distance from initial point of contact of two streams
y = distance from center line

Greek Letters

η = dimensionless similarity variable
λ = U_2/U_1 = ratio of free-stream velocities
ψ = stream function
ρ = fluid density
μ = Newtonian viscosity

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The Equation of Convective Diffusion and Its Solution in the Small Penetration Approximation

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Because the diffusion coefficient in a liquid is very small, the depth of penetration by diffusion is also very small. It is possible therefore to use in numerous cases the small penetration approximation, that is, to solve the problems of mass transfer by means of the solutions valid for a semiinfinite liquid (*1, 2*). The aim of the present communication is to examine in the framework of this approximation the mass transfer through a free-moving interface. The equation of convective diffusion for a liquid having a free-moving interface may be solved exactly by means of a similarity transformation, as is illustrated by the problem of the absorption of a gas in a thin liquid film in wave motion along a vertical wall.

The equation for a moving interface between a liquid and a gas is, with respect to a fixed-coordinate system xOy,

$$y = H\,(x, t) \tag{1}$$

Since we want to know the mass flux at the free interface, it is natural to write the convective-diffusion equation in a frame of reference bound to this interface. Such a coordinates system may be defined, for instance, by the distance y_1 to the interface and by the corresponding distance x_1 along the interface.

When the depth of penetration by diffusion is small, we may use for the velocity components in the convective-diffusion equation expressions valid in the vicinity of the interface:

$$u_1 \approx (u_1)_{y_1=0} = u_{10} \tag{2}$$

$$v_1 \approx (v_1)_{y_1=0} + \left(\frac{\partial v_1}{\partial y_1}\right)_{y_1=0} \cdot y_1 = \left(\frac{\partial v_1}{\partial y_1}\right)_{y_1=0} y_1 \tag{3}$$

($(v_1)_{y_1=0} = 0$ since the frame of reference is bound to the interface, which is a streamline.)

Approximating also $\nabla^2 c \approx (\partial^2 c/\partial y_1^2)$ and taking into account that in the region of interest $y_1 << R$, the equation of convective diffusion in the small penetration approximation may be written as

$$\frac{\partial c}{\partial t} + u_{10}\frac{\partial c}{\partial x_1} + \left(\frac{\partial v_1}{\partial y_1}\right)_{y_1=0} y_1 \frac{\partial c}{\partial y_1} = D\frac{\partial^2 c}{\partial y_1^2} \quad (4)$$

This equation must be solved for the boundary conditions

$$\begin{aligned} c &= c_0 \quad \text{for} \quad y_1 = 0 \\ c &= 0 \quad \text{for} \quad y_1 = \infty \end{aligned} \quad (5)$$

Equation (4) and the boundary conditions (5) are compatible with a solution of the form

$$\frac{c}{c_0} = f(\eta), \quad \eta = \frac{y_1}{\delta}, \quad \delta = \delta(x_1, t) \quad (6)$$

Indeed, use of the new variable η transforms Equation (4) into

$$-\left[\delta\frac{\partial \delta}{\partial t} + u_{10}\delta\frac{\partial \delta}{\partial x_1} - \left(\frac{\partial v_1}{\partial y_1}\right)_{y_1=0}\delta^2\right]\eta\frac{dc}{d\eta} = D\frac{d^2c}{d\eta^2} \quad (7)$$

In order that $c/c_0 = f(\eta)$ and $\delta = \delta(x_1, t)$, we must have

$$\delta\frac{\partial \delta}{\partial t} + u_{10}\delta\frac{\partial \delta}{\partial x_1} - \left(\frac{\partial v_1}{\partial y_1}\right)_{y_1=0}\delta^2 = \text{const.}\, D \quad (8)$$

The thickness δ may be obtained by solving this equation (linear in δ^2) with partial derivatives and taking into account that

$\delta \rightarrow$ finite limit for $x_1 = 0$.

For the concentration one obtains

$$\frac{d^2c}{d\eta^2} + \text{const.}\,\eta\frac{dc}{d\eta} = 0 \quad (9)$$

For the case of wave motion, we will use for u the equation

$$u = 3\bar{u}\,[1 + (\beta - 1)\alpha\sin k(x - wt) - (\beta - 1)\alpha^2\sin^2 k(x - wt)]\left(\frac{y}{h} - \frac{y^2}{2h^2}\right) \quad (10)$$

established by Kapitza (3).

Since the wave length is large compared with the film thickness, one may take $x_1 \approx x$ and $y_1 \approx h - y$.

Using (10) and also the change of variable $z = x_1 - wt$, we can write Equation (4) as

$$-\left(\frac{2}{3}\beta - Z\right)\frac{\partial c}{\partial z} + Z\frac{\partial c}{\partial x_1} - y_1\frac{dZ}{dz}\frac{\partial c}{\partial y_1} = \frac{2}{3}\frac{D}{\bar{u}}\frac{\partial^2 c}{\partial y_1^2} \quad (11)$$

where

$$Z \equiv 1 + (\beta - 1)\alpha\sin kz - (\beta - 1)\alpha^2\sin^2 kz \quad (12)$$

To solve Equation (11) for the boundary conditions (5), we must look for solutions of the form

$$\frac{c}{c_0} = f(\eta), \quad \eta \equiv \frac{y_1}{\delta}, \quad \delta = \delta(x_1, z)$$

Equation (11) becomes

$$-\left[-\left(\frac{2}{3}\beta - Z\right)\delta\frac{\partial \delta}{\partial z} + Z\delta\frac{\partial \delta}{\partial x_1} + \delta^2\frac{dZ}{dz}\right]\eta\frac{dc}{d\eta} = \frac{2D}{3\bar{u}}\frac{d^2c}{d\eta^2} \quad (13)$$

In order that $c/c_0 = f(\eta)$ and $\delta = \delta(x_1, z)$, we must have

$$-\left(\frac{2}{3}\beta - Z\right)\delta\frac{\partial \delta}{\partial z} + Z\delta\frac{\partial \delta}{\partial x_1} + \delta^2\frac{dZ}{dz} = \frac{40D}{9\bar{u}}\epsilon^2 = \text{const} \quad (14)$$

where ϵ is an arbitrary numerical constant ($\epsilon \neq 0$). A positive value must be selected for the constant (const.) in order that the boundary conditions (5) be satisfied.

When Equation (14) is taken into account, (13) leads to

$$\frac{d^2c}{d\eta^2} + \frac{20}{3}\epsilon^2\eta\frac{dc}{d\eta} = 0 \quad (15)$$

Equation (15) must be solved for the boundary conditions

$$\frac{c}{c_0} = 1 \text{ for } \eta = 0, \; c = 0, \text{ for } \eta \rightarrow \infty \quad (16)$$

Denoting

$$\Delta \equiv \left(\frac{\delta}{\epsilon}\right)^2$$

Equation (13) may be written as

$$-\left(\frac{2}{3}\beta - Z\right)\frac{\partial \Delta}{\partial z} + Z\frac{\partial \Delta}{\partial x_1} + 2\frac{dZ}{dz}\Delta = \frac{80D}{9\bar{u}} \quad (17)$$

The general solution of this equation has the form

$$\frac{9\bar{u}k}{80D}\left(\frac{2}{3}\beta - Z\right)^2\Delta + k\int_0^z\left(\frac{2\beta}{3} - Z\right)dz = \phi(\theta) \quad (18)$$

where ϕ is an arbitrary function of the argument

$$\theta(x_1, z) \equiv kx_1 + k\int_0^z \frac{Z\,dz}{\frac{2}{3}\beta - Z} \quad (19)$$

The quantity δ signifying a depth of penetration by diffusion, it is natural to have

$$\delta = 0 \quad \text{for} \quad x_1 = 0 \quad (20)$$

The form of the function ϕ may be determined by using condition (20). This condition lets us obtain from Equation (18)

$$\phi(\theta) = k\int_0^z\left(\frac{2}{3}\beta - Z\right)dz, \quad x_1 = 0 \quad (21)$$

The curve ϕ vs. θ may be traced graphically by observing that to a value of the dimensionless variable kz corresponds a value of ϕ given by Equation (21) and a value of θ given by Equation (19) (in which x_1 is nil). In this manner the function Δ is completely determined.

Equation (15) and the boundary conditions lead to

$$\frac{c}{c_0} = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{\frac{10}{3}}\epsilon\eta} e^{-\xi^2}\,d\xi \equiv \text{erfc}\left(\frac{y_1}{\sqrt{\frac{3\Delta}{10}}}\right) \quad (23)$$

For the mass flux at the free surface, one obtains

$$N = -D\left(\frac{\partial c}{\partial y_1}\right)_{y_1=0} = \frac{2\,Dc_0}{\sqrt{\Delta}}\cdot\sqrt{\frac{10}{3\pi}} \quad (24)$$

The results obtained by the above exact method of analysis do not differ practically from those obtained for the same problem by means of the polynomial approximation (4);* therefore the curve ϕ vs. θ and also some numerical results concerning the mass flux may be taken from our previous paper. The essential difference between the present and the previous paper consists in the fact that here exact solutions are given without postulating that c/c_0 may be written as a polynomial of the fourth degree with respect to y_1/δ and without using the approximations implied by the integral method.

* The ratio of the mass flux obtained by means of the exact method and of the polynominal method is $\sqrt{10/3\pi}$.

The method applied here has been used for the analysis of several other cases, namely mass transfer between a spherical drop and the continuous phase, mass transfer into a turbulent liquid (5), and mass transfer between a bubble and a pulsed liquid (6). In all these cases the convective-diffusion equation has form (4) and the similarity variable $\eta = y_1/[\delta(x_1, t)]$ enables it to be reduced to an ordinary differential equation for the concentration c and to a first-order partial-differential equation for $\delta = \delta(x_1, t)$.

NOTATION

c = concentration
c_0 = value of c_0 for $y_1 = 0$
D = diffusion coefficient
h = film thickness
k = wave number
R = curvature radius
t = time
u = x component of velocity
u_1 = x_1 component of velocity
v_1 = y_1 component of velocity
w = wave velocity
z = $x_1 - wt$
Z = function defined by Equation (12)
x, y = coordinates in a fixed system of reference; distance along the wall and distance to the wall in the case of a thin liquid film in wave motion
x_1 = coordinate along the interface
y_1 = distance to the interface
α = ratio of the wave amplitude to the average value of h
β = ratio of propagation velocity to the average velocity $\bar{u}$
δ = penetration depth by diffusion
Δ $\equiv (\delta/\epsilon)^2$
ϵ = arbitrary numerical constant
η = y_1/δ

Subscript and Superscript

o = the values of the quantities at $y_1 = 0$
$-$ = time and y average values

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Film-Penetration Models for Mass Transfer with Chemical Reaction

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The film-penetration model for mass transfer was developed by Toor and Marchello (6) on the assumption that the average thickness of the liquid element had a finite value, L. Toor and Marchello showed that the film (7), penetration (2), and surface-renewal (1) models were merely limiting conditions of their film-penetration model. Huang and Kuo (3) revised the film-penetration theory to include a first-order or pseudofirst-order chemical reaction. Two further applications of the film-penetration theory are presented here to describe mass transfer through a surface resistance with chemical reaction and mass transfer with chemical reaction in spherical drops.

MODEL I—MASS TRANSFER THROUGH A SURFACE RESISTANCE WITH CHEMICAL REACTION

The following model was postulated to describe the mass-transfer process across a place surface with a surface resistance by applying the film-penetration concept.

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2} - KC \tag{1}$$

$$C = C_o; \quad x \geqq L; \quad t > 0$$
$$C = C_o; \quad x \geqq 0; \quad t = 0 \tag{2}$$

$$-D\left(\frac{\partial C}{\partial x}\right)_{x=0} = k_s\,(C_i - C); \; x = 0; \; t > 0 \tag{3}$$

Equation (3) is the surface-resistance condition which is used instead of surface equilibrium because it is nearer the actual situation encountered in mass transfer operations. The quantity $(C_i - C)$ is the departure from surface equilibrium due to the surface resistance.

The average rate of mass transfer over the entire surface is found by combining the rate for surfaces of age t with the age distribution function and by integrating over all ages:

$$R = \int_0^\infty \psi(t)\,\phi(t)\,dt \tag{4}$$

For this situation these functions are described as

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